Extended Abstract for **Network Games with Incomplete Information**

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Introduction

This paper studies the role of incomplete information when individuals face a game where externalities, in the form of strategic complementarities or substitutabilities, depend on a network structure that determines linkages in a group.

Consider, for example, the case of a classroom. There is empirical evidence that suggests that peer effects are not homogeneous and depend on the precise geometry of relations among students. Recent theoretical contributions on the analysis of games played in networks provide simple models that build a very precise bridge between the geometry of interactions and individual effort provision when the nature of peer effects, measured as the direct complementarities in effort for any pair of friends, and the direct returns from effort are perfectly known. Building upon this literature, we relax the assumption of perfect knowledge in complementarties and returns, and study a more realistic situation where individuals have imperfect information about these fundamentals. We get a full-fledged characterization of equilibrium behavior in this incomplete information game and derive comparative statics results with possible policy implications.

The model

We consider a finite group of agents, that we denote by $\mathcal{N} = \{1, \ldots, n\}$ linked by social relations. We assume that social relations are *undirected*, meaning that if Alice is connected with Bob, then Bob is also connected with Alice. We also assume that social relations are *unweighted*, meaning that all relevant information from these relations is whether two agents know each other or not; there are no intensities in social relations. Social connections are gathered in an adjacency matrix $\mathbf{G} = (g_{ij})$ of a *social network*, where $g_{ij} = 1$ if agents i and j are connected, and $g_{ij} = 0$ otherwise.

We build upon a complete information game with network externalities that has captured a lot of attention in the recent literature of games played on networks. In the complete information setting, agents have to choose an effort $x_i \geq 0$ and utilities are linear-quadratic and depend on the geometry of the social network:

$$u_i\left(x_i, \mathbf{x}_{-i}; \mathbf{G}\right) = \alpha x_i - \frac{1}{2} x_i^2 + \beta \sum_{j \in N} g_{ij} x_i x_j \tag{1}$$

The parameter α naturally captures direct returns on effort. And since

$$\beta = \frac{\partial^2 u_i}{\partial x_i \partial x_j} \qquad \text{if } g_{ij} = 1$$

the parameter β captures the intensity of strategic complementarities, if $\beta > 0$, or substitutabilities, if $\beta < 0$, in actions of linked individuals.

We depart from the assumption of complete information in returns and complementarities. We assume that individuals have only partially informative signals about them. The game becomes a Bayesian game with common values. To make the analysis more transparent, we analyze in the main text the case of *incomplete* information on α and *complete* information on β. There is a strong resemblance in intuitions and in the technical analysis when there is incomplete information on the strength of strategic complementarities.

To allow for some generality while keeping tractability, we assume that the number of possible states of states of the world, in this case the strength of returns to individual activity, is finite. The group shares a common prior and, conditional on the state, individuals receive independent signals about it. When the number of possible signals is finite, as we assume, the information structure can be subsumed in what we define as the *information matrix* $\mathbf{\Gamma}$, where each entry γ_{ij} is the conditional probability that, given two signal realizations, the value of the second one is s_j when the first one is s_i.

Equilibrium Analysis

To characterize the equilibrium, we rely on centrality measures developed in social network analysis. In particular, our main tool is the Bonacich centrality measure, defined as follows: given a network $\mathbf{G}$ and a discount factor δ, and a vector of weights $\mathbf{w}$, the vector of weighted Bonacich centralities is given by

$$b_i(\delta; \mathbf{G}, \mathbf{w}) = w_i + \delta \sum_{j \neq i} g_{ij} b_j(\delta; \mathbf{G}, \mathbf{w})$$

In words, the Bonacich centtrality vector is an endogenous measure of relevance where an individual has some initial level of exogenous centrality w_i and the rest of his centrality depends, according to the parameter δ, to the centrality of his neighbours in the network. A similar idea underlies, for example, the PageRank algorithm that Google uses to rank webpages.

We characterize conditions that ensure there is a unique Bayesian equilibrium of our game. The analysis highlights the role of the two matrices, $\mathbf{G}$ and $\mathbf{\Gamma}$, in equilibrium behavior. In particular, the spectral properties of both matrices fully determine the shape of individual equilibrium actions. We get the following result: the strategy of individual i, that maps the different signals $s_1, \ldots, s_m$ to actions $x_i(s_1), \ldots, x_i(s_m)$, can be written in the following way

$$x_i(s_j) = w_{ij}^1 b_i(\mu_1 \beta; \mathbf{G}, \mathbf{1}) + \cdots + w_{ij}^k b_i(\mu_k \beta; \mathbf{G}, \mathbf{1})$$

where $\mu_1, \ldots, \mu_k$ are the k different eigenvalues of the information matrix $\mathbf{\Gamma}$, and the weights w_{ij}^l only depend on the eigenvectors of this same matrix.[1] This result shows that the relation between equilibrium activity and centrality in Ballester *et al* (2006) and following papers[2] is preserved, in a particular way, when moving to settings with incomplete information. The novelty of this characterization is that we can keep track of the effects of incomplete information in a very precise way, according to spectral properties of the information matrix.

The trick to derive the equilibrium characterization above is to rewrite the equilibrium conditions as a function of the Kronecker product of the adjancency and the information matrices, $\mathbf{G} \otimes \mathbf{\Gamma}$, and exploit the good properties of the Kronecker product.

The eigenvalues of the information matrix have some interpretation: for example, the second largest eigenvalue of the matrix preovides information about the speed of convergence of iterated expectations of the value of the signal a particular individual receives. In this sense, there is some connection between the model we study here and a recent literature of belief formation in networks (see for example, Golub and Jackson, 2010, and references therein).

Comparative Statics and Policy Implications

We can exploit the structure of the equilibrium described above in different directions. First, we study the question of which individual is more important when the group tries to maximize aggregate activity. The question of finding the key player in the network has a very clear answer in the case of complete information (see Ballester et al, 2006). We show that introducing incomplete information can distort the solution: the individual that is more relevant in the complete information case does not need to necessarily be the same one as in the case of incomplete information.

[1] We get a precise close-form expression of equilibrium strategies. We avoid here entering in too much detail.

[2] See, for example, Ballester and Calvo-Armengol (2010), Bramoulle and Kranton (2007) or Bramoulle *et al* (2010)

A second question we try to address is the following one: if the social planner had the ability to increase the precision of the information available to the group, which groups would benefit more of this according to actual precision and geometry of the network? For example, think about two different classrooms with different network geometries (that due to local socioeconomic conditions may reflect segregation patterns) and with different quality of information (one of them may be a classroom from a neighbourhood where families are motivated and aware of the value of education, while the other one is a classroom in a connflictive neighbourhood, where information about the benefits of education is much more noisy). To deal with this question, we have to play with particular information structures that allows us to give meaning to the notion of quality/precision of information. For a natural family of information structures where we can do this exercise, we find a neat characterization of the effect of a bit more of information in social welfare. We get a close-form expression of the increase of social welfare that depends again on a vector of Bonacich centrality measures, but in this case of the vector $\mathbf{b}(\delta; \mathbf{G}, \mathbf{b}(\delta; \mathbf{G}, \mathbf{1}))$, i.e. on an iterated Bonacich centrality: the weighted Bonacich centrality when the initial vector of weights is the unweighted Bonacich centrality.

One conclusion from the analysis of this last question is that, usually, groups that already have more information and that are more connected obtain more benefits from the increase in the quality of information. This is because equilibrium behavior and welfare have increasing returns to information quality, and the strength of this increasing returns is larger when the group is more connected (this is reminiscent of the fact that the Bonacich centrality measure is convex on the discount factor, and in equilibrium strategies the discount factors in the different Bonacich centralities depend on the eigenvalues of the information matrix that, precisely, capture the quality of information available to the group).

References

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